TURNPIKE PROPERTY FOR EXTREMALS
OF VARIATIONAL PROBLEMS
WITH VECTOR-VALUED FUNCTIONS

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ABSTRACT. In this paper we study the structure of extremals $\nu\colon [0,T] \to R^n$ of variational problems with large enough T, fixed end points and an integrand f from a complete metric space of functions. We will establish the turnpike property for a generic integrand f. Namely, we will show that for a generic integrand f, any small $\varepsilon > 0$ and an extremal $\nu\colon [0,T] \to R^n$ of the variational problem with large enough T, fixed end points and the integrand f, for each $\tau \in [L_1, T - L_1]$ the set $\{\nu(t)\colon t \in [\tau, \tau + L_2]\}$ is equal to a set $H(f)$ up to ε in the Hausdorff metric. Here $H(f) \subset R^n$ is a compact set depending only on the integrand f and $L_1 > L_2 > 0$ are constants which depend only on ε and $|\nu(0)|, |\nu(T)|$.

1. INTRODUCTION

In this paper we analyse the structure of optimal solutions of the variational problem

(P) $$\int_0^T f(z(t), z'(t)) \, dt \to \min, \quad z(0) = x, \; z(T) = y,$$

$z\colon [0,T] \to R^n$ is an absolutely continuous function

where $T > 0$, $x, y \in R^n$ and $f\colon R^{2n} \to R^1$ is an integrand.

An optimal solution $\nu\colon [0,T] \to R^n$ of the variational problem (P) always depends on the integrand f and on x, y, T. We say that the integrand f has the *turnpike property* if for large enough T the dependence on x, y, T is not essential. Namely, for any $\varepsilon > 0$ there exist constants $L_1 > L_2 > 0$ which depend only on $|x|$, $|y|$, ε such that for each $\tau \in [L_1, T - L_1]$ the set

$$\{\nu(t)\colon t \in [\tau, \tau + L_2]\}$$

is equal to a set $H(f)$ up to ε in the Hausdorff metric where $H(f) \subset R^n$ is a compact set depending only on the integrand f.

More formally we say that an integrand $f = f(x, u) \in C(R^{2n})$ has the *turnpike property* if there exists a compact set $H(f) \subset R^n$ such that for each bounded set $K \subset R^n$ and each $\varepsilon > 0$ there exist numbers $L_1 > L_2 > 0$ such that for each

$T \geq 2L_1$, each $x, y \in K$ and an optimal solution $\nu \colon [0, T] \to R^n$ for the variational problem (P) the relation

$$\text{dist}(H(f), \{\nu(t) \colon t \in [\tau, \tau + L_2]\}) \leq \varepsilon$$

holds for each $\tau \in [L_1, T - L_1]$. (Here $\text{dist}(\cdot, \cdot)$ is the Hausdorff metric.)

The turnpike property is well known in mathematical economics. It was studied by many authors for optimal trajectories of a von Neumann-Gale model determined by a superlinear set-valued mapping (see Makarov and Rubinov [14] and the survey [16]) and for optimal trajectories of convex autonomous systems (see Carlson, Haurie and Leizarowitz [7, Ch. 4.6].) In the control theory the turnpike property was established by Artstein and Leizarowitz [1] for a tracking periodic problem. In all these cases we have an optimal control problem with a convex cost function and a convex set of trajectories. Asymptotic turnpike properties for optimal control problems with infinite time horizon were studied by Brock and Haurie [4], Carlson [5], Carlson, Haurie and Jabrane [6], Leizarowitz [10] and Zaslavski [19].

Our goal is to show that the turnpike property is a general phenomenon which holds for a large class of variational problems with vector-valued functions. We consider the complete metric space of integrands $\overline{\mathfrak{M}}_k$ (k is a nonnegative integer) described below and establish the existence of a set $\mathcal{F} \subset \overline{\mathfrak{M}}_k$ which is a countable intersection of open everywhere dense sets in $\overline{\mathfrak{M}}_k$ and such that each integrand $f \in \mathcal{F}$ has the turnpike property.

Moreover we show that the turnpike property holds for approximate solutions of variational problems with a generic integrand f and that the turnpike phenomenon is stable under small perturbations of a generic integrand f.

A better understanding of the general nature of the turnpike phenomenon was achieved by our recent study of discrete-time control systems [17, 18] for which we established a weak version of the turnpike property. More recently in Zaslavski [20] employing the reduction to finite rewards by Leizarowitz [11, 12] and the representation formula by Leizarowitz and Mizel [13] an analogous result was established for optimal solutions of the variational problem (P) with $x, y \in R^n$, large enough T and a generic integrand f belonging to the space of functions $\mathfrak{A}$ described below.

In the weak version of the turnpike property established in [20] for an optimal solution of the problem (P) with $x, y \in R^n$, large enough T and a generic integrand $f \in \mathfrak{A}$ the relation

$$\text{dist}(H(f), \{\nu(t) \colon t \in [\tau, \tau + L_2]\}) \leq \varepsilon$$

with L_2 which depends on ε and $|x|$, $|y|$ and a compact set $H(f) \subset R^n$ depending only on the integrand f, holds for each $\tau \in [0, T] \setminus E$ where $E \subset [0, T]$ is a measurable subset such that the Lebesgue measure of E does not exceed a constant which depends on ε and $|x|$, $|y|$.

The turnpike property which will be established in the present work guarantees that we may take $E = [0, L_1] \cup [T - L_1, T]$ where $L_1 > 0$ is a constant which depends on ε and $|x|$, $|y|$.

Denote by $|\cdot|$ the Euclidean norm in R^n and denote by $\mathfrak{A}$ the set of continuous functions $f \colon R^n \times R^n \to R^1$ which satisfy the following assumptions:

(A)(i) *for each $x \in R^n$ the function $f(x, \cdot) \colon R^n \to R^1$ is convex;*

(ii) *$f(x, u) \geq \sup\{\psi(|x|), \psi(|u|)|u|\} - a$ for each $(x, u) \in R^n \times R^n$ where $a > 0$ is a constant and $\psi \colon [0, \infty) \to [0, \infty)$ is an increasing function such that $\psi(t) \to +\infty$ as $t \to \infty$ (here a and ψ are independent on f);*

(iii) *for each $M, \varepsilon > 0$ there exist $\Gamma, \delta > 0$ such that*

$$|f(x_1, u_1) - f(x_2, u_2)| \leq \varepsilon \sup\{f(x_1, u_1), f(x_2, u_2)\}$$

for each $u_1, u_2, x_1, x_2 \in R^n$ which satisfy

$$|x_i| \leq M, \ |u_i| \geq \Gamma \ (i = 1, 2), \quad \sup\{|x_1 - x_2|, |u_1 - u_2|\} \leq \delta.$$

It is an elementary exercise to show that an integrand $f = f(x, u) \in C^1(R^{2n})$ belongs to $\mathfrak{A}$ if f satisfies assumptions (Ai), (Aii) with a constant $a > 0$ and a function $\psi \colon [0, \infty) \to [0, \infty)$ and there exists an increasing function $\psi_0 \colon [0, \infty) \to [0, \infty)$ such that for each $x, u \in R^n$

$$\sup\{|\partial f/\partial x(x, u)|, |\partial f/\partial u(x, u)|\} \leq \psi_0(|x|)(1 + \psi(|u|)|u|).$$

For the set $\mathfrak{A}$ we consider the uniformity which is determined by the following base

$$E(N, \varepsilon, \lambda) = \{(f, g) \in \mathfrak{A} \times \mathfrak{A} \colon |f(x, u) - g(x, u)| \leq \varepsilon \ (u, x \in R^n, |x|, |u| \leq N),$$
$$(|f(x, u)| + 1)(|g(x, u)| + 1)^{-1} \in [\lambda^{-1}, \lambda] \ (x, u \in R^n, |x| \leq N)\}$$

where $N > 0$, $\varepsilon > 0$, $\lambda > 1$ (see Kelley [9]).

It was shown in Zaslavski [20] that the uniform space $\mathfrak{A}$ is metrizable and complete. We consider functionals of the form

$$(1.1) \qquad\qquad I^f(T_1, T_2, x) = \int_{T_1}^{T_2} f(x(t), x'(t)) \, dt$$

where $f \in \mathfrak{A}$, $0 \leq T_1 \leq T_2 < +\infty$ and $x \colon [T_1, T_2] \to R^n$ is an absolutely continuous (a.c.) function.

For $f \in \mathfrak{A}$, $y, z \in R^n$ and numbers T_1, T_2 satisfying $0 \leq T_1 < T_2$ we set

$$(1.2) \quad U^f(T_1, T_2, y, z) = \inf\{I^f(T_1, T_2, x) \colon x \colon [T_1, T_2] \to R^n \text{ is an a.c. function}$$
$$\text{satisfying } x(T_1) = y, \ x(T_2) = z\}.$$

It is easy to see that $-\infty < U^f(T_1, T_2, y, z) < +\infty$ for each $f \in \mathfrak{A}$, each $y, z \in R^n$ and all numbers T_1, T_2 satisfying $0 \leq T_1 < T_2$.

Let $f \in \mathfrak{A}$. For any a.c. function $x \colon [0, \infty) \to R^n$ we set

$$(1.3) \qquad\qquad J(x) = \liminf_{T \to \infty} T^{-1} I^f(0, T, x).$$

Of special interest is the *minimal long-run average cost growth rate*

$$(1.4) \qquad \mu(f) = \inf\{J(x) \colon x \colon [0, \infty) \to R^n \text{ is an a.c. function}\}.$$

Clearly $-\infty < \mu(f) < +\infty$. By a simple modification of the proof of Proposition 4.4 in Leizarowitz and Mizel [13] (see [20, Theorems 8.1, 8.2]) we established the representation formula

$$(1.5) \qquad U^f(0, T, x, y) = T\mu(f) + \pi^f(x) - \pi^f(y) + \theta_T^f(x, y),$$
$$x, y \in R^n, \quad T \in (0, \infty),$$

where $\pi^f \colon R^n \to R^1$ is a continuous function and $(T, x, y) \to \theta_T^f(x, y) \in R^1$ is a continuous nonnegative function defined for $T > 0$, $x, y \in R^n$,

$$(1.6) \quad \pi^f(x) = \inf \left\{ \liminf_{T \to +\infty} [I^f(0, T, \nu) - \mu(f)T] \colon \nu \colon [0, \infty) \to R^n \right.$$

$$\left. \text{is an a.c. function satisfying } \nu(0) = x \right\}, \quad x \in R^n,$$

and for every $T > 0$, every $x \in R^n$ there is $y \in R^n$ satisfying $\theta_T^f(x, y) = 0$.

Here we follow Leizarowitz [11] in defining "good functions" for the infinite horizon variational problem with the integrand f.

An a.c. function $x \colon [0, \infty) \to R^n$ is called an (f)-*good function* if the function $\Phi_x^f \colon T \to I^f(0, T, x) - \mu(f)T$, $T \in (0, \infty)$ is bounded. In [20] we showed that for each $f \in \mathfrak{A}$ and each $z \in R^n$ there exists an (f)-good function $\nu \colon [0, \infty) \to R^n$ satisfying $\nu(0) = z$.

Propositions 1.1 and 3.2 in Zaslavski [20] imply the following result.

Proposition 1.1. *For any a.c. function $x \colon [0, \infty) \to R^n$ either*

$$I^f(0, T, x) - T\mu(f) \to +\infty \text{ as } T \to \infty$$

or

$$\sup\{|I^f(0, T, x) - T\mu(f)| \colon T \in (0, \infty)\} < \infty.$$

Moreover any (f)-good function $x \colon [0, \infty) \to R^n$ is bounded.

We denote $d(x, B) = \inf\{|x - y| \colon y \in B\}$ for $x \in R^n$, $B \subset R^n$. Denote by $\mathrm{dist}(A, B)$ the Hausdorff metric for two sets $A \subset R^n$, $B \subset R^n$. For every bounded a.c. function $x \colon [0, \infty) \to R^n$ define

$$(1.7) \quad \Omega(x) = \{y \in R^n \colon \text{there exists a sequence } \{t_i\}_{i=0}^{\infty} \subset (0, \infty) \text{ for which}$$

$$t_i \to \infty, \; x(t_i) \to y \quad \text{as} \quad i \to \infty\}.$$

We say that an integrand $f \in \mathfrak{A}$ has Property B if $\Omega(\nu_2) = \Omega(\nu_1)$ for all (f)-good functions $\nu_i \colon [0, \infty) \to R^n$, $i = 1, 2$.

In Zaslavski [20, Theorem 2.1] we establish the following result which describes the limit behaviour of (f)-good functions for a generic $f \in \mathfrak{A}$.

Theorem 1.1. *There exists a set $\mathcal{F} \subset \mathfrak{A}$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{A}$ and such that each $f \in \mathcal{F}$ has Property B.*

By Proposition 1.1 for each integrand $f \in \mathfrak{A}$ which has Property B there exists a compact set $H(f) \subset R^n$ such that $\Omega(\nu) = H(f)$ for each (f)-good function $\nu \colon [0, \infty) \to R^n$.

Denote by $\mathfrak{M}$ the set of all functions $f \in C^1(R^{2n})$ satisfying the following assumptions which ensure that each solution of (P) belongs to $C^2([0, T]; R^n)$:

$$\partial f / \partial u_i \in C^1(R^{2n}) \quad \text{for } i = 1, \ldots, n;$$

the matrix $(\partial^2 f / \partial u_i \partial u_j)(x, u)$, $i, j = 1, \ldots, n$, is positive definite for all $(x, u) \in R^{2n}$;

$$f(x, u) \geq \sup\{\psi(|x|), \psi(|u|)|u|\} - a \quad \text{for all } (x, u) \in R^n \times R^n;$$

there exist a number $c_0 > 1$ and monotone increasing functions $\phi_i \colon [0, \infty) \to [0, \infty)$, $i = 0, 1, 2$, such that

$$\phi_0(t)t^{-1} \to +\infty \text{ as } t \to +\infty, \ \ f(x, u) \geq \phi_0(c_0|u|) - \phi_1(|x|), \quad x, u \in R^n;$$

$$\sup\{|\partial f/\partial x_i(x, u)|, |\partial f/\partial u_i(x, u)|\} \leq \phi_2(|x|)(1 + \phi_0(|u|)),$$

$$x, u \in R^n, \ i = 1, \ldots, n.$$

It is easy to see that $\mathfrak{M} \subset \mathfrak{A}$. We will establish the following result.

Theorem 1.2. *Assume that an integrand $f \in \mathfrak{M}$ has Property B and $\varepsilon, K > 0$. Then there exists a neighborhood $\mathcal{U}$ of f in $\mathfrak{A}$ and numbers $M > K$, $l_0 > l > 0$, $\delta > 0$ such that for each $g \in \mathcal{U}$, each $T \geq 2l_0$ and each a.c. function $\nu \colon [0, T] \to R^n$ which satisfies*

$$|\nu(0)|, |\nu(T)| \leq K, \quad I^g(0, T, \nu) \leq U^g(0, T, \nu(0), \nu(T)) + \delta$$

the relation $|\nu(t)| \leq M$ holds for all $t \in [0, T]$ and

(1.8) $$\text{dist}(H(f), \{\nu(t) \colon t \in [\tau, \tau + l]\}) \leq \varepsilon$$

for each $\tau \in [l_0, T - l_0]$. Moreover if $d(\nu(0), H(f)) \leq \delta$, then (1.8) holds for each $\tau \in [0, T - l_0]$ and if $d(\nu(T), H(f)) \leq \delta$, then (1.8) holds for each $\tau \in [l_0, T - l]$.

Let $k \geq 1$ be an integer. Denote by $\mathfrak{A}_k$ the set of all integrands $f \in \mathfrak{A} \cap C^k(R^{2n})$. For $p = (p_1, \ldots, p_{2n}) \in \{0, \ldots, k\}^{2n}$ and $f \in C^k(R^{2n})$ we set

$$|p| = \sum_{i=1}^{2n} p_i, \quad D^p f = \partial^{|p|} f/\partial y_1^{p_1} \ldots \partial y_{2n}^{p_{2n}}.$$

For the set $\mathfrak{A}_k$ we consider the uniformity which is determined by the following base.

$$E(N, \varepsilon, \lambda) = \{(f, g) \in \mathfrak{A}_k \times \mathfrak{A}_k \colon |D^p f(x, u) - D^p g(x, u)| \leq \varepsilon$$
$$(u, x \in R^n, |x|, |u| \leq N, p \in \{0, \ldots, k\}^{2n}, |p| \leq k),$$
$$|f(x, u) - g(x, u)| \leq \varepsilon \ (u, x \in R^n, |x|, |u| \leq N),$$
$$(|f(x, u)| + 1)(|g(x, u)| + 1)^{-1} \in [\lambda^{-1}, \lambda](x, u \in R^n, |x| \leq N)\}$$

where $N > 0$, $\varepsilon > 0$, $\lambda > 1$ (see Kelley [9]). It is easy to verify that the uniform space $\mathfrak{A}_k$ is metrizable and complete (see [20], Section 2).

For each integer $k \geq 1$ we define $\mathfrak{M}_k = \mathfrak{M} \cap \mathfrak{A}_k$. Set

$$\mathfrak{A}_0 = \mathfrak{A}, \quad \mathfrak{M}_0 = \mathfrak{M}.$$

Let $k \geq 0$ be an integer. Denote by $\overline{\mathfrak{M}}_k$ the closure of $\mathfrak{M}_k$ in $\mathfrak{A}_k$ and consider the topological subspace $\overline{\mathfrak{M}}_k \subset \mathfrak{A}_k$ with the relative topology. We will establish the following result.

Theorem 1.3. *Let $q \geq 0$ be an integer. Then there exists a set $\mathcal{F}_q \subset \overline{\mathfrak{M}}_q$ which is a countable intersection of open everywhere dense subsets of $\overline{\mathfrak{M}}_q$ and such that each $f \in \mathcal{F}_q$ has Property B and the following property:*

For each $\varepsilon, K > 0$ there exist a neighborhood $\mathcal{U}$ of f in $\mathfrak{A}$ and numbers $M > K$, $l_0 > l > 0$, $\delta > 0$ such that for each $g \in \mathcal{U}$, each $T \geq 2l_0$ and each a.c. function $\nu \colon [0, T] \to R^n$ which satisfies

$$|\nu(0)|, \ |\nu(t)| \leq K, \quad I^g(0, T, \nu) \leq U^g(0, T, \nu(0), \nu(T)) + \delta$$

the relation $|\nu(t)| \leq M$ holds for all $t \in [0, T]$ and

$$(1.9) \qquad\qquad \mathrm{dist}(H(f), \{\nu(t) \colon t \in [\tau, \tau + l]\}) \leq \varepsilon$$

for each $\tau \in [l_0, T - l_0]$. Moreover if $d(\nu(0), H(f)) \leq \delta$, then (1.9) holds for each $\tau \in [0, T - l_0]$ and if $d(\nu(T), H(f)) \leq \delta$, then (1.9) holds for each $\tau \in [l_0, T - l]$.

Theorems 1.2 and 1.3 are extensions to the class of variational problems with vector-valued functions of the main result in Zaslavski [21] established for a class of one-dimensional variational problems arising in continuum mechanics which was discussed in Leizarowitz and Mizel [13] and Coleman, Marcus and Mizel [8]. In the approach used in [21] the following property of this class of one-dimensional variational problems established in Leizarowitz and Mizel [13] played a crucial role.

Property C. In the space of integrands there exists an everywhere dense subset E such that for each $f \in E$ there exists an (f)-good periodic trajectory.

It is not clear whether Property C holds in general. In Zaslavski [20] and in the present paper we develop a more general approach based on the idea that the validity of Property B implies the weak version of the turnpike property for an integrand $f \in \mathfrak{A}$ and implies the turnpike property for an integrand $f \in \mathfrak{M}$.

2. Auxiliary results

In [20] we established the following results.

Proposition 2.1 ([20, Proposition 3.1]). *For each $f \in \mathfrak{A}$ there exists a neighborhood $\mathcal{U}$ of f in $\mathfrak{A}$ and a number $M > 0$ such that for each $g \in \mathcal{U}$ and each (g)-good function $x \colon [0, \infty) \to R^n$*

$$\limsup_{t \to \infty} |x(t)| < M.$$

Proposition 2.2 ([20, Proposition 3.2]). *Let $f \in \mathfrak{A}$ and $M_1, M_2, c > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathfrak{A}$ and $S > 0$ such that for each $g \in \mathcal{U}$, each $T_1 \in [0, \infty)$ and each $T_2 \in [T_1 + c, \infty)$ the following property holds.*

For each $x, y \in R^n$ satisfying $|x|, |y| \leq M_1$ and each a.c. function $\nu \colon [T_1, T_2] \to R^n$ satisfying $\nu(T_1) = x$, $\nu(T_2) = y$, $I^g(T_1, T_2, \nu) \leq U^g(T_1, T_2, x, y) + M_2$ the following relation holds: $|\nu(t)| \leq S$ $(t \in [T_1, T_2])$.

Proposition 2.3 ([20, Proposition 3.8]). *Let $f \in \mathfrak{A}$, $0 < c_1 < c_2 < \infty$, $D, \varepsilon > 0$. Then there is a neighborhood V of f in $\mathfrak{A}$ such that for each $g \in V$, each $T_1, T_2 \geq 0$ satisfying $T_2 - T_1 \in [c_1, c_2]$ and each a.c. function $x \colon [T_1, T_2] \to R^n$ satisfying $\inf\{I^f(T_1, T_2, x), I^g(T_1, T_2, x)\} \leq D$ the relation $|I^f(T_1, T_2, x) - I^g(T_1, T_2, x)| \leq \varepsilon$ holds.*

Proposition 2.4 ([20, Proposition 3.9]). *Let $f \in \mathfrak{A}$, $0 < c_1 < c_2 < \infty$, $c_3, \varepsilon > 0$. Then there exists a neighborhood V of f in $\mathfrak{A}$ such that for each $g \in V$, each $T_1, T_2 \geq 0$ satisfying $T_2 - T_1 \in [c_1, c_2]$ and each $z, y \in R^n$ satisfying $|y|, |z| \leq c_3$ the relation $|U^f(T_1, T_2, y, z) - U^g(T_1, T_2, y, z)| \leq \varepsilon$ holds.*

Proposition 2.5 ([20, Theorem 5.1]). *Assume that $f \in \mathfrak{A}$ and there exists a compact set $H(f) \subset R^n$ such that $\Omega(\nu) = H(f)$ for each (f)-good function $\nu \colon [0, \infty) \to R^n$. Let ε be a positive number. Then there exists an integer $L \geq 1$ such that for each (f)-good function $\nu \colon [0, \infty) \to R^n$*

$$\mathrm{dist}(H(f), \{\nu(t) \colon t \in [T, T + L]\}) \leq \varepsilon \quad \textit{for all large } T.$$

Proposition 2.6 ([20, Theorem 6.1])**.** *Assume that* $f \in \mathfrak{A}$*.   Then the mapping* $(T_1, T_2, x, y) \to U^f(T_1, T_2, x, y)$ *is continuous for* $T_1 \in [0, \infty)$*,* $T_2 \in (T_1, \infty)$*,* $x, y \in R^n$*.*

Proposition 2.7 ([20, Theorem 2.3])**.** *Assume that* $f \in \mathfrak{A}$ *and there exists a compact set* $H(f) \subset R^n$ *such that* $\Omega(\nu) = H(f)$ *for each* (f)*-good function* $\nu \colon [0, \infty) \to R^n$*. Let* ε *be a positive number. Then there exist an integer* $L \geq 1$ *and a neighborhood* $\mathcal{U}$ *of* f *in* $\mathfrak{A}$ *such that for each* $g \in \mathcal{U}$ *and each* (g)*-good function* $\nu \colon [0, \infty) \to R^n$

$$\mathrm{dist}(H(f), \{\nu(t) \colon t \in [T, T + L]\}) \leq \varepsilon \quad \textit{for all large } T.$$

Proposition 2.8 ([20, Lemma 10.2])**.** *Assume that* $f \in \mathfrak{A}$ *and* $H(f) \subset R^n$ *is a compact set such that* $\Omega(\nu) = H(f)$ *for each* (f)*-good function* $\nu \colon [0, \infty) \to R^n$*. Let* $\varepsilon_0 \in (0, 1)$*,* $K_0, M_0 > 0$ *and let* l *be a positive integer such that for each* (f)*-good function* $x \colon [0, \infty) \to R^n$

$$\mathrm{dist}(H(f), \{x(t) \colon t \in [T, T + l]\}) \leq 8^{-1}\varepsilon_0$$

for all large T *(the existence of* l *follows from Proposition 2.5). Then there exist an integer* $N \geq 10$ *and a neighborhood* $\mathcal{U}$ *of* f *in* $\mathfrak{A}$ *such that for each* $g \in \mathcal{U}$*, each* $S \in [0, \infty)$ *and each a.c. function* $x \colon [S, S + Nl] \to R^n$ *satisfying*

$$|x(S)|, \ |x(S + Nl)| \leq K_0,$$
$$I^g(S, S + Nl, x) \leq U^g(S, S + Nl, x(S), x(S + Nl)) + M_0$$

there exists an integer $i_0 \in [0, N - 8]$ *such that for all* $T \in [S + i_0 l, S + (i_0 + 7)l]$

$$\mathrm{dist}(H(f), \{x(t) \colon t \in [T, T + l]\}) \leq \varepsilon_0.$$

Proposition 2.9 ([20, Lemma 10.3])**.** *Assume that* $f \in \mathfrak{A}$ *and* $H(f) \subset R^n$ *is a compact set such that* $\Omega(\nu) = H(f)$ *for each* (f)*-good function* $\nu \colon [0, \infty) \to R^n$*. Let* $\varepsilon > 0$*. Then there exists* $\delta > 0$ *such that for each* $x_1, x_2 \in R^n$ *which satisfy* $d(x_i, H(f)) \leq \delta$*,* $i = 1, 2$ *there exists an a.c. function* $\nu \colon [0, T] \to R^n$ *for which*

$$T \geq 1, \ \nu(0) = x_1, \ \nu(T) = x_2, \quad I^f(0, T, \nu) - \pi^f(x_1) + \pi^f(x_2) - T\mu(f) \leq \varepsilon.$$

Proposition 2.10 ([20, Lemma 10.4])**.** *Assume that* $f \in \mathfrak{A}$ *and* $H(f) \subset R^n$ *is a compact set such that* $\Omega(\nu) = H(f)$ *for each* (f)*-good function* $\nu \colon [0, \infty) \to R^n$*. Let* $\varepsilon \in (0, 1)$ *and let* L *be a positive integer such that for each* (f)*-good function* $\nu \colon [0, \infty) \to R^n$

$$(2.1) \qquad\qquad \mathrm{dist}(H(f), \{\nu(t) \colon t \in [S, S + L]\}) \leq \varepsilon$$

for all large S *(the existence of* L *follows from Proposition 2.5).*

   Then there exists $\delta > 0$ *such that for each* $T \in [L, \infty)$ *and each a.c. function* $\nu \colon [0, T] \to R^n$ *which satisfies*

$$d(\nu(0), H(f)) \leq \delta, \ d(\nu(T), H(f)) \leq \delta,$$
$$I^f(0, T, \nu) - T\mu(f) - \pi^f(\nu(0)) + \pi^f(\nu(T)) \leq \delta$$

relation (2.1) holds for every $S \in [0, T - L]$*.*

Proposition 2.11 ([20, Lemma 9.1])**.** *Assume that* $f \in \mathfrak{A}$*. Then there exists a compact set* $H^* \subset R^n$ *which has the following properties:*

   there exists an (f)*-good function* $u \colon [0, \infty) \to R^n$ *such that* $\Omega(u) = H^*$*;*

   for each (f)*-good function* $\nu \colon [0, \infty) \to R^n$ *either* $\Omega(\nu) = H^*$ *or* $\Omega(\nu) \backslash H^* \neq \varnothing$*.*

Proposition 2.12 ([20, Lemmas 9.3, 9.4]). *Let $f \in \mathfrak{A}$ and let H^* be as guaranteed in Proposition 2.11. Assume that $\phi\colon R^n \to [0,\infty)$ is a continuous bounded function such that $H^* = \{x \in R^n\colon \phi(x) = 0\}$. For $r \in (0,1]$ we set*

$$f_r(x, u) = f(x, u) + r\phi(x), \ \ x, u \in R^n.$$

Then $f_r \in \mathfrak{A}$, $r \in (0,1]$ and for each $r \in (0,1]$ and each (f_r)-good function $\nu\colon [0,\infty) \to R^n$

$$\Omega(\nu) = H^*.$$

Moreover for any neighborhood $\mathcal{U}$ of f in $\mathfrak{A}$ there exists $r_0 \in (0,1)$ such that $f_r \in \mathfrak{A}$ for every $r \in (0, r_0)$.

Proposition 2.13 ([20, Proposition 3.4]). *Assume that $f \in \mathfrak{A}$, $M_1 > 0$, $0 \leq T_1 < T_2$, $x_i\colon [T_1, T_2] \to R^n$, $i = 1, 2, \ldots$ is a sequence of a.c. functions such that $I^f(T_1, T_2, x_i) \leq M_1$, $i = 1, 2, \ldots$. Then there exist a subsequence $\{x_{i_k}\}_{k=1}^\infty$ and an a.c. function $x\colon [T_1, T_2] \to R^n$ such that $I^f(T_1, T_2, x) \leq M_1$, $x_{i_k}(t) \to x(t)$ as $k \to \infty$ uniformly in $[T_1, T_2]$ and $x'_{i_k} \to x'$ as $k \to \infty$ weakly in $L^1(R^n; (T_1, T_2))$.*

Theorem 8.1, (8.2) and Proposition 7.3 in [20] imply the following result.

Proposition 2.14. *Let $f \in \mathfrak{A}$. Then $\pi^f(x) \to +\infty$ as $|x| \to \infty$.*

The following result was established in Aubin and Ekeland [2, Ch. 2, Sec. 3].

Proposition 2.15. *Let Ω be a closed subset of R^q. Then there exists a bounded nonnegative function $\phi \in C^\infty(R^q)$ such that $\Omega = \{x \in R^q\colon \phi(x) = 0\}$ and for each sequence of nonnegative integers $p_1, \ldots, p_q$ the function $\partial^{|p|}\phi/\partial x_1^{p_1}, \ldots, \partial x_q^{p_q}\colon R^q \to R^1$ is bounded where $|p| = \sum_{i=1}^q p_i$.*

We can establish the following proposition which is a higher dimensional version of a well known result (Ball and Mizel [3], Morrey [15]).

Proposition 2.16. *Suppose that $f \in \mathfrak{M}$, $x, y \in R^n$, $T_1 \in [0, \infty)$, $T_2 > T_1$ and $w\colon [T_1, T_2] \to R^n$ is an a.c. function such that*

$$w(T_1) = x, \quad w(T_2) = y, \quad I^f(T_1, T_2, w) = U^f(T_1, T_2, x, y).$$

Then $w \in C^2([T_1, T_2]; R^n)$.

3. Structure of the proof of Theorem 1.2

Assume that $f \in \mathfrak{M}$ and $H(f) \subset R^n$ is a compact set such that $\Omega(\nu) = H(f)$ for each (f)-good function $\nu\colon [0, \infty) \to R^n$.

We will describe briefly the proof of Lemma 4.4 which is established in Section 4 and which plays a crucial role in our discussion of Theorem 1.2.

For each a.c. function $u\colon [\tau_1, \tau_2] \to R^n$ where $\tau_1 \geq 0$, $\tau_2 > \tau_1$ and each $r_1, r_2 \in [\tau_1, \tau_2]$ satisfying $r_1 < r_2$ we set

$$\sigma(r_1, r_2, u) = I^f(r_1, r_2, u) - \pi^f(u(r_1)) + \pi^f(u(r_2)) - (r_2 - r_1)\mu(f).$$

Let $\varepsilon > 0$. To prove Lemma 4.4 we need to show that there is a number $q \geq 8$ such that for each $h_1, h_2 \in H(f)$ there exists an a.c. function $\nu\colon [0, q] \to R^n$ which satisfies

(3.1) $\nu(0) = h_1, \quad \nu(q) = h_2, \quad \sigma(0, q, \nu) \leq \varepsilon.$

By Proposition 2.6 there exists a sequence of positive numbers $\{\delta_i\}_{i=0}^\infty$ such that

$$(3.2) \qquad \delta_0 \in (0, 8^{-1}\varepsilon), \quad \delta_{i+1} < \delta_i, \quad i = 0, 1, \ldots .$$

and for each integer $i \geq 0$, each $x_1, x_2, y_1, y_2 \in H(f)$ which satisfy $|x_j - y_j| \leq \delta_i$, $j = 1, 2$ the following relations hold:

$$(3.3) \qquad \begin{aligned} |U^f(0, 1, x_1, x_2) - U^f(0, 1, y_1, y_2)| &\leq 2^{-i-8}\varepsilon, \\ |\pi^f(x_j) - \pi^f(y_j)| &\leq 2^{-i-8}\varepsilon, \quad j = 1, 2. \end{aligned}$$

We will show that there exists an (f)-good function $\nu_* \colon [0, \infty) \to H(f)$ such that $\sigma(T_1, T_2, \nu_*) = 0$ for each $T_1 \geq 0$, $T_2 > T_1$. Then we will define a function $\phi \colon [0, \infty) \to R^1$ as

$$(3.4) \qquad \phi(\tau) = I^f(0, 1, \nu_* + P_\tau) - \mu(f) - \pi^f(\nu_*(0)) + \pi^f(\nu_*(\tau)), \quad \tau \in [0, \infty),$$

where

$$(3.5) \qquad P_\tau(t) = t(\nu_*(\tau) - \nu_*(1)), \quad t \in R^1, \quad \tau \in [0, \infty),$$

and verify that

$$(3.6) \qquad \phi \in C^1([0, \infty); R^1), \ \phi(1) = 0, \ \phi(t) \geq 0, \ t \in [0, \infty).$$

We can find a number $L \geq 10$ and a sequence of numbers $\{T_p\}_{p=1}^\infty$ such that

$$(3.7) \qquad \mathrm{dist}(H(f), \{\nu_*(t) \colon t \in [T, T+L]\}) \leq 4^{-1}\delta_0 \text{ for all } T \in [0, \infty),$$

$$(3.8) \qquad T_p \geq 2L + 8, \ |\nu_*(0) - \nu_*(T_p)| \leq 2^{-8}\delta_p, \quad p = 1, 2, \ldots .$$

Fix a positive number $\varepsilon_0 < 2^{-8}L^{-1}\varepsilon$. By (3.6) there exists a positive number Δ such that

$$(3.9) \qquad \Delta < 2^{-8}, \ |\phi'(t)| \leq 2^{-1}\varepsilon_0, \ t \in [1-\Delta, 1+\Delta].$$

Choose an integer $N > 64(L+1)\Delta^{-1}$ and set $q = \sum_{i=1}^N T_i + 8L + 8$.

Let $h_1, h_1 \in H(f)$. We will construct an a.c. function $\nu \colon [0, q] \to R^n$ satisfying (3.1). By the definition of L there exists numbers t_1, t_2 such that

$$(3.10) \qquad t_1 \in [0, L], \ t_2 \in [8, L+8], \ |h_j - \nu_*(t_j)| \leq 4^{-1}\delta_0, \quad j = 1, 2.$$

We set $\Delta_0 = (N-1)^{-1}(8L+8-(t_2-t_1))$ and verify that $\Delta_0 \in (0, \Delta)$. By using (3.8), (3.10) and the definition of $\{\delta_i\}_{i=0}^\infty$ we can construct functions $w_0 \colon [0, T_1 - t_1] \to R^n$ and $u_0 \colon [0, t_2] \to R^n$ such that

$$(3.11) \qquad \begin{aligned} w_0(0) &= h_1, \quad w_0(T_1 - t_1) = \nu_*(0), \quad \sigma(0, T_1 - t_1, w_0) \leq 2^{-6}\varepsilon, \\ u_0(0) &= \nu_*(0), \quad u_0(t_2) = h_2, \quad \sigma(0, t_2, u_0) \leq 2^{-7}\varepsilon. \end{aligned}$$

For each integer $k \geq 1$ there exists an a.c. function $w_k \colon [0, \Delta_0 + T_{k+1}] \to R^n$ such that

$$w_k(t) = \nu_*(t) + P_{1-\Delta_0}(t), \ t \in [0, 1], \quad w_k(t) = \nu_*(t - \Delta_0), \ t \in [1, \Delta_0 + T_{k+1} - 1],$$
$$w_k(\Delta_0 + T_{k+1}) = \nu_*(0),$$

$$I^f(\Delta_0 + T_{k+1} - 1, \Delta_0 + T_{k+1}, w_k) = U^f(0, 1, w_k(\Delta_0 + T_{k+1} - 1), w_k(\Delta_0 + T_{k+1})).$$

By using (3.4)–(3.6), (3.9), (3.8) and the definition of $\{\delta_i\}_{i=0}^\infty$ we show that

$$\sigma(0, T_{k+1} + \Delta_0, w_k) \leq 2^{-1}\varepsilon_0\Delta_0 + 2^{-k-8}\varepsilon, \qquad k = 1, 2, \ldots .$$

We will finally define $\nu \colon [0, q] \to R^n$ as a concatenation of the functions w_k, $k = 0, \ldots, N-1$, u_0 and show that (3.1) holds.

The structure of the proof of Theorem 1.2. For simplicity we will only sketch the proof of the turnpike property for the integrand f and will not discuss the stability of the turnpike phenomenon under small perturbations of f. We choose a small enough number $\delta > 0$ and large enough numbers $l_0 > l > 0$ depending on ε, K.

Assume that $T \geq 2l_0$ and an a.c. function $\nu\colon [0,T] \to R^n$ satisfies

$$(3.12) \qquad |\nu(0)|,\ |\nu(T)| \leq K, \quad I^f(0,T,\nu) \leq U^f(0,T,\nu(0),\nu(T)) + \delta.$$

We will show that for each $\tau \in [l_0, T - l_0]$

$$(3.13) \qquad \operatorname{dist}(H(f), \{\nu(t)\colon t \in [\tau, \tau + l]\}) \leq \varepsilon.$$

Assume the contrary. Then there is a number $\tau \in [l_0, T - l_0]$ for which

$$(3.14) \qquad \operatorname{dist}(H(f), \{\nu(t)\colon t \in [\tau, \tau + l]\}) > \varepsilon.$$

By Proposition 2.8 there are numbers $S_1, S_2 \in [0, T]$ such that

$$(3.15) \qquad d(\nu(S_i), H(f)) \leq \delta, \quad i = 1, 2, \quad S_2 - \tau,\ \tau - S_1 \in [c_1, c_2],$$

where c_1, c_2 are some positive constants depending on ε, K.

It follows from (3.14), (3.15) and Proposition 2.10 that

$$(3.16) \qquad\qquad\qquad \sigma(S_1, S_2, \nu) > \delta_0$$

where $\delta_0 > 8\delta$ is some constant depending on ε, K. By using (3.15) and Lemma 4.4 we show that there exists an a.c. function $u\colon [0, T] \to R^n$ such that

$$(3.17) \qquad u(t) = \nu(t), \quad t \in [0, S_1] \cup [S_2, T], \quad \sigma(S_1, S_2, u) < \delta_0 - \delta.$$

It follows from (3.12), (3.16), (3.17) that

$$\delta \geq I^f(0, T, \nu) - I^f(0, T, u) = \sigma(S_1, S_2, \nu) - \sigma(S_1, S_2, u) > \delta.$$

The obtained contradiction proves that (3.13) holds for each $\tau \in [l_0, T - l_0]$.

4. Proof of Theorem 1.2

Assume that $f \in \mathfrak{M}$ and $H(f) \subset R^n$ is a compact set such that $\Omega(\nu) = H(f)$ for each (f)-good function $\nu\colon [0, \infty) \to R^n$.

Lemma 4.1. *Let $h \in H(f)$. Then there exists an (f)-good function $\nu\colon [0, \infty) \to H(f)$ such that $\nu(0) = h$ and*

$$(4.1) \qquad I^f(T_1, T_2, \nu) = \mu(f)(T_2 - T_1) + \pi^f(\nu(T_1)) - \pi^f(\nu(T_2))$$

for each $T_1 \geq 0$, $T_2 > T_1$.

Proof. Consider any (f)-good function $w\colon [0, \infty) \to R^n$. Then

$$\Omega(w) = H(f).$$

By Proposition 2.1 the function w is bounded. It is easy to see that the following property holds:

(a) for each $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that for each $T_1 \geq T(\varepsilon)$, $T_2 > T_1$

$$I^f(T_1, T_2, w) - \mu(f)(T_2 - T_1) - \pi^f(w(T_1)) + \pi^f(w(T_2)) \leq \varepsilon.$$

There exists a sequence of numbers $\{T_p\}_{p=0}^{\infty} \subset [0, \infty)$ such that

$$(4.2) \qquad T_{p+1} \geq T_p + 1, \quad p = 0, 1, \ldots, \quad w(T_p) \to h \quad \text{as} \quad p \to \infty.$$

For every integer $p \geq 1$ we set

$$(4.3) \qquad\qquad \nu_p(t) = w(t + T_p), \quad t \in [0, \infty).$$

By Proposition 2.13, the boundness of w, (4.3) and property (a) there exists a subsequence $\{\nu_{p_j}\}_{j=1}^{\infty}$ and an a.c. function $\nu\colon [0,\infty) \to R^n$ such that for each integer $N \geq 1$

(4.4)
$$\nu_{p_j}(t) \to \nu(t) \quad \text{as} \quad j \to \infty \text{ uniformly in } [0,N],$$
$$I^f(0,N,\nu) \leq \liminf_{j\to\infty} I^f(0,N,\nu_{p_j}).$$

(4.2)–(4.4) imply that $\nu(0) = h$ and $\nu(t) \in H(f)$, $t \in [0,\infty)$. It follows from property (a) and (4.3), (4.4) that (4.1) holds for each $T_1 \geq 0$, $T_2 > T_1$. The lemma is proved.

By Lemma 4.1 there exists an (f)-good function $\nu_*\colon [0,\infty) \to H(f)$ such that

(4.5) $$I^f(T_1,T_2,\nu_*) = \mu(f)(T_2 - T_1) + \pi^f(\nu_*(T_1)) - \pi^f(\nu_*(T_2))$$

for each $T_1 \geq 0$, $T_2 > T_1$.

It follows from Proposition 2.16 that

(4.6) $$\nu_* \in C^2([0,\infty); R^n).$$

Lemma 4.2. *The function* $\pi^f \cdot \nu_* \in C^1([0,\infty); R^1)$.

Proof. By (4.5) for each $T \geq 0$
$$\pi^f(\nu_*(T)) = -I^f(0,T,\nu_*) + \mu(f)T + \pi^f(\nu_*(0)).$$

Together with (4.6) this implies the assertion of the lemma.
For each $\tau \in [0,\infty)$ we define

(4.7) $$P_\tau(t) = t(\nu_*(\tau) - \nu_*(1)), \quad t \in R^1, \quad \psi(\tau) = I^f(0,1,\nu_* + P_\tau).$$

Lemma 4.3. $\psi \in C^1([0,\infty); R^1)$.

Proof. For $\lambda, t \in [0,\infty)$ we set

(4.8) $$B(\lambda,t) = (\nu_*(t) + P_\lambda(t), \nu_*'(t) + P_\lambda'(t)).$$

Let $\tau, h \in [0,\infty)$, $\tau \neq h$ and $t \in [0,1]$. By (4.7), (4.8) there exists $\lambda_h(t) \in [\inf\{h,\tau\}, \sup\{h,\tau\}]$ such that

$$(h-\tau)^{-1}[f(B(h,t)) - f(B(\tau,t))] = \partial f/\partial x(B(\lambda_h(t),t))t\nu_*'(\lambda_h(t))$$
$$+ \partial f/\partial u(B(\lambda_h(t),t))\nu_*'(\lambda_h(t)) \to \partial f/\partial x(B(\tau,t))t\nu_*'(\tau) + \partial f/\partial u(B(\tau,t))\nu_*'(\tau)$$

as $h \to \tau$ uniformly for all $t \in [0,1]$. This implies that $\psi \in C^1([0,\infty); R^1)$. The lemma is proved.

Lemma 4.4. *Let* $\varepsilon > 0$. *Then there exists a number* $q \geq 8$ *such that for each* $h_1, h_2 \in H(f)$ *there exists an a.c. function* $\nu\colon [0,q] \to R^n$ *which satisfies*

(4.9) $$\nu(0) = h_1, \quad v(q) = h_2,$$

(4.10) $$I^f(0,q,\nu) \leq q\mu(f) + \pi^f(\nu(0)) - \pi^f(\nu(q)) + \varepsilon.$$

Proof. Define a function $\phi\colon [0,\infty) \to R^1$ as follows:

(4.11) $$\phi(t) = \psi(t) - \mu(f) - \pi^f(\nu_*(0)) + \pi^f(\nu_*(t)), \quad t \in [0,\infty).$$

It follows from (4.11), (4.7), Lemmas 4.2, 4.3, (4.5) and the representation formula (see (1.5), (1.6)) that

(4.12) $$\phi \in C^1([0,\infty); R^1), \quad \phi(1) = 0, \quad \phi(t) \geq 0, \quad t \in [0,\infty).$$

By Proposition 2.6 there exists a sequence of positive numbers $\{\delta_i\}_{i=0}^{\infty}$ such that

$$(4.13) \qquad \delta_0 \in (0, 8^{-1}\varepsilon), \quad \delta_{i+1} < \delta_i, \quad i = 0, 1, \dots .$$

and for each integer $i \geq 0$, each $x_1, x_2, y_1, y_2 \in H(f)$ which satisfy $|x_j - y_j| \leq \delta_i$, $j = 1, 2$ the following relations hold:

$$(4.14) \qquad \begin{aligned} |U^f(0, 1, x_1, x_2) - U^f(0, 1, y_1, y_2)| &\leq 2^{-i-8}\varepsilon, \\ |\pi^f(x_j) - \pi^f(y_j)| &\leq 2^{-i-8}\varepsilon, \quad j = 1, 2. \end{aligned}$$

By the definition of ν, Proposition 2.5 there exists an integer $L \geq 10$ such that

$$(4.15) \qquad \operatorname{dist}(H(f), \{\nu_*(t)\colon t \in [T, T + L]\} \leq 4^{-1}\delta_0$$

for all $T \in [0, \infty)$.

Since $\Omega(\nu_*) = H(f)$ and $\nu_*(0) \in H(f)$ there exists a sequence of numbers $\{T_p\}_{p=1}^{\infty}$ such that

$$(4.16) \qquad T_p \geq 2L + 8, \quad |\nu_*(0) - \nu_*(T_p)| \leq 2^{-8}\delta_p, \quad p = 1, 2, \dots .$$

Fix a positive number ε_0 for which

$$(4.17) \qquad \varepsilon_0 < 2^{-8}L^{-1}\varepsilon.$$

It follows from (4.12) that there exists a positive number Δ such that

$$(4.18) \qquad \Delta < 2^{-8}, \quad |\phi'(t)| \leq 2^{-1}\varepsilon_0, \quad t \in [1 - \Delta, 1 + \Delta].$$

Choose an integer

$$(4.19) \qquad N > 64(L + 1)\Delta^{-1}$$

and set

$$(4.20) \qquad q = \sum_{i=1}^{N} T_i + 8L + 8.$$

Let $h_1, h_1 \in H(f)$. We will construct an a.c. function $\nu\colon [0, q] \to R^n$ satisfying (4.9), (4.10). It follows from (4.15) which holds for each $T \in [0, \infty)$ that there exists numbers t_1, t_2 such that

$$(4.21) \qquad t_1 \in [0, L], \quad t_2 \in [8, L + 8], \quad |h_j - \nu_*(t_j)| \leq 4^{-1}\delta_0, \quad j = 1, 2.$$

Set

$$(4.22) \qquad \Delta_0 = (N - 1)^{-1}(8L + 8 - (t_2 - t_1)).$$

(4.22), (4.21), (4.19), (4.18) imply that

$$(4.23) \qquad \Delta_0 \in (0, \Delta).$$

For each a.c. function $u\colon [\tau_1, \tau_2] \to R^n$ where $\tau_1 \geq 0$, $\tau_2 > \tau_1$ and each $r_1, r_2 \in [\tau_1, \tau_2]$ satisfying $r_1 < r_2$ we set

$$(4.24) \qquad \sigma(r_1, r_2, u) = I^f(r_1, r_2, u) - \pi^f(u(r_1)) + \pi^f(u(r_2)) - (r_2 - r_1)\mu(f).$$

It follows from (4.16), (4.21) and Proposition 2.13 that there exists an a.c. function $w_0\colon [0, T_1 - t_1] \to R^n$ such that

$$(4.25)$$
$$w_0(0) = h_1, \quad w_0(t) = \nu_*(t_1 + t), \quad t \in [1, T_1 - t_1 - 1], \quad w_0(T_1 - t_1) = \nu_*(0),$$
$$I^f(\tau, \tau + 1, w_0) = U^f(0, 1, w_0(\tau), w_0(\tau + 1)), \qquad \tau = 0, T_1 - t_1 - 1.$$

By (4.24), the definition of ν_*, (4.5), (4.25), (4.21), (4.16) and the definition of $\{\delta_j\}_{j=0}^{\infty}$

$$
\begin{aligned}
\sigma(&0, T_1 - t_1, w_0) \\
&= \sigma(0, 1, w_0) + \sigma(T_1 - t_1 - 1, T_1 - t_1, w_0) \\
&= U^f(0, 1, h_1, \nu_*(t_1 + 1)) - \pi^f(h_1) + \pi^f(\nu_*(t_1 + 1)) - \mu(f) \\
&\quad + U^f(0, 1, \nu_*(T_1 - 1), \nu_*(0)) - \pi^f(\nu_*(T_1 - 1)) + \pi^f(\nu_*(0)) - \mu(f) \\
&\leq 4 \cdot 2^{-8}\varepsilon + U^f(0, 1, \nu_*(t_1), \nu_*(t_1 + 1)) - \pi^f(\nu_*(t_1)) \\
&\quad + \pi^f(\nu_*(t_1 + 1)) - \mu(f) + U^f(0, 1, \nu_*(T_1 - 1), \nu_*(T_1)) \\
&\quad - \pi^f(\nu_*(T_1 - 1)) + \pi^f(\nu_*(T_1)) - \mu(f) \leq 2^{-6}\varepsilon.
\end{aligned}
$$

(4.26)

Let $k \geq 1$ be an integer. By (4.16), (4.7), (4.23), (4.18) and Proposition 2.13 there exists an a.c. function $w_k \colon [0, \Delta_0 + T_{k+1}] \to R^n$ such that

(4.27)
$$
\begin{aligned}
w_k(t) &= \nu_*(t) + P_{1-\Delta_0}(t), \ t \in [0, 1], \ w_k(t) = \nu_*(t - \Delta_0), \ t \in [1, \Delta_0 + T_{k+1} - 1], \\
&\quad w_k(\Delta_0 + T_{k+1}) = \nu_*(0), \\
I^f(&\Delta_0 + T_{k+1} - 1, \Delta_0 + T_{k+1}, w_k) = U^f(0, 1, w_k(\Delta_0 + T_{k+1} - 1), w_k(\Delta_0 + T_{k+1})).
\end{aligned}
$$

By (4.27), (4.7)

(4.28)
$$
w_k(0) = \nu_*(0).
$$

We will estimate $\sigma(0, T_{k+1} + \Delta_0, w_k)$. It follows from (4.27), (4.24), (4.5) that

(4.29) $\quad \sigma(0, T_{k+1} + \Delta_0, w_k) = \sigma(0, 1, w_k) + \sigma(T_{k+1} + \Delta_0 - 1, T_{k+1} + \Delta_0, w_k).$

(4.27), (4.24), (4.7), (4.11) imply that

(4.30)
$$
\sigma(0, 1, w_k) = \phi(1 - \Delta_0).
$$

It follows from (4.30), (4.23), (4.18), (4.12) that

(4.31)
$$
\sigma(0, 1, w_k) \leq 2^{-1}\Delta_0 \varepsilon_0.
$$

By (4.27), (4.24), (4.16), the definition of ν_*, (4.5), the definition of $\{\delta_i\}_{i=0}^{\infty}$ (see (4.13), (4.14))

$$
\begin{aligned}
\sigma(&T_{k+1} + \Delta_0 - 1, T_{k+1} + \Delta_0, w_k) \\
&= U^f(0, 1, \nu_*(T_{k+1} - 1), \nu_*(0)) - \pi^f(\nu_*(T_{k+1} - 1)) \\
&\quad + \pi^f(\nu_*(0)) - \mu(f) \\
&\leq U^f(0, 1, \nu_*(T_{k+1} - 1), \nu_*(T_{k+1})) - \pi^f(\nu_*(T_{k+1} - 1)) \\
&\quad + \pi^f(\nu_*(T_{k+1})) - \mu(f) + 2 \cdot 2^{-k-9}\varepsilon = 2^{-k-8}\varepsilon.
\end{aligned}
$$

(4.32)

Combining (4.29), (4.31), (4.32) we obtain that

(4.33)
$$
\sigma(0, T_{k+1} + \Delta_0, w_k) \leq 2^{-1}\varepsilon_0 \Delta_0 + 2^{-k-8}\varepsilon.
$$

By Proposition 2.13 there exists an a.c. function $u_0 \colon [0, t_2] \to R^n$ such that

(4.34)
$$
\begin{aligned}
&u_0(t) = \nu_*(t), \ t \in [0, t_2 - 1], \ u_0(t_2) = h_2, \\
&I^f(t_2 - 1, t_2, u_0) = U^f(0, 1, u_0(t_2 - 1), u_0(t_2)).
\end{aligned}
$$

It follows from (4.34), (4.24), (4.21), the definition of $\{\delta_i\}_{i=0}^{\infty}$ (see (4.13), (4.14)), the definition of ν_*, (4.5) that

$$
\begin{aligned}
\sigma(0, t_2, u_0) &= \sigma(t_2 - 1, t_2, u_0) \\
&= U^f(0, 1, \nu_*(t_2 - 1), h_2) - \pi^f(\nu_*(t_2 - 1)) + \pi^f(h_2) - \mu(f) \\
&\leq U^f(0, 1, \nu_*(t_2 - 1), \nu_*(t_2)) - \pi^f(\nu_*(t_2 - 1)) \\
&\quad + \pi^f(\nu_*(t_2)) - \mu(f) + 2^{-7}\varepsilon \leq 2^{-7}\varepsilon.
\end{aligned}
\tag{4.35}
$$

(4.22), (4.20) imply that

$$
T_1 - t_1 + \sum_{k=1}^{N-1}(\Delta_0 + T_{k+1}) + t_2 = q.
\tag{4.36}
$$

By (4.36), (4.25), (4.27), (4.28), (4.34) there exists an a.c. function $\nu \colon [0, q] \to R^n$ such that

$$
\nu(t) = w_0(t),\ t \in [0, T_1 - t_1], \quad \nu(t) = w_k\left(t - \left(\sum_{i=1}^{k} T_i + (k-1)\Delta_0 - t_1\right)\right),
$$

$$
t \in \left[\sum_{i=1}^{k} T_i + (k-1)\Delta_0 - t_1, \sum_{i=1}^{k+1} T_i + k\Delta_0 - t_1\right], \quad k = 1, \dots, N-1,
\tag{4.37}
$$

$$
\nu(t) = u_0\left(t - \left(\sum_{i=1}^{N} T_i + (N-1)\Delta_0 - t_1\right)\right), \quad t \in \left[\sum_{i=1}^{N} T_i + (N-1)\Delta_0 - t_1, q\right].
$$

(4.37), (4.25), (4.36), (4.34) imply that

$$
\nu(0) = h_1, \quad \nu(q) = h_2.
$$

It follows from (4.24), (4.37), (4.26), (4.33), (4.35), (4.22), (4.21), (4.17) that

$$
\begin{aligned}
&I^f(0, q, \nu) - \pi^f(\nu(0)) + \pi^f(\nu(q)) - q\mu(f) \\
&\qquad = \sigma(0, T_1 - t_1, w_0) + \sum_{k=1}^{N-1} \sigma(0, T_{k+1} + \Delta_0, w_k) + \sigma(0, t_2, u_0) \\
&\qquad \leq 2^{-6}\varepsilon + \sum_{k=1}^{N-1}(2^{-1}\varepsilon_0\Delta_0 + 2^{-k-8}\varepsilon) + 2^{-7}\varepsilon \leq 2^{-5}\varepsilon + 2^{-1}(N-1)\varepsilon_0\Delta_0 \\
&\qquad \leq 2^{-5}\varepsilon + 2^{-1}(9L + 16)\varepsilon_0 \leq 2^{-1}\varepsilon.
\end{aligned}
$$

This completes the proof of the lemma.

Proof of Theorem 1.2. Let $\varepsilon, K > 0$. We may assume that

$$
\varepsilon < 1, \quad K > \sup\{|h| \colon h \in H(f)\} + 4.
$$

By Proposition 2.2 there exist a neighborhood $\mathcal{U}_1$ of f in $\mathfrak{A}$ and a number $M > K$ such that for each $g \in \mathcal{U}_1$, each $T_1 \geq 0$, $T_2 \geq T_1 + 1$ and each a.c. function $\nu \colon [T_1, T_2] \to R^n$ which satisfies

$$
|\nu(T_i)| \leq 2K + 4,\ i = 1, 2, \quad I^g(T_1, T_2, \nu) \leq U^g(T_1, T_2, \nu(T_1), \nu(T_2)) + 2
\tag{4.38}
$$

the following relation holds:

$$
|\nu(t)| \leq M\ (t \in [T_1, T_2]).
\tag{4.39}
$$

By Proposition 2.6 there exists

(4.40) $$\delta_1 \in (0, 8^{-1}\varepsilon)$$

such that for each $x_1, x_2, y_1, y_2 \in R^n$ which satisfy

(4.41) $\quad |x_i|, |y_i| \le 2M + 4 + 2\sup\{|h|\colon h \in H(f)\}, \quad |x_i - y_i| \le 4\delta_1, \quad i = 1, 2$

the following relations hold:

(4.42)
$$|U^f(0, 1, x_1, x_2) - U^f(0, 1, y_1, y_2)| \le 2^{-8}\varepsilon,$$
$$|\pi^f(x_i) - \pi^f(y_i)| \le 2^{-8}\varepsilon, \quad i = 1, 2.$$

By Proposition 2.5 there exists an integer $l \ge 1$ such that for each (f)-good function $\nu\colon [0, \infty) \to R^n$

(4.43) $$\operatorname{dist}(H(f), \{\nu(t)\colon t \in [T, T + l]\}) \le \varepsilon$$

for all large T. By Proposition 2.10 there exists

(4.44) $$\delta_0 \in (0, 2^{-1}\delta_1)$$

such that for each $T \in [l, \infty)$ and each a.c. function $\nu\colon [0, T] \to R^n$ which satisfies

(4.45) $$d(\nu(0), H(f)) \le \delta_0, \quad d(\nu(T), H(f)) \le \delta_0,$$

(4.46) $$I^f(0, T, \nu) - T\mu(f) - \pi^f(\nu(0)) + \pi^f(\nu(T)) \le \delta_0$$

the relation

(4.47) $$\operatorname{dist}(H(f), \{\nu(t)\colon t \in [S, S + l]\}) \le \varepsilon$$

holds for every $S \in [0, T - l]$. By Proposition 2.6 there exists

(4.48) $$\delta \in (0, 32^{-1}\delta_0)$$

such that for each $x_1, x_2, y_1, y_2 \in R^n$ which satisfy

(4.49) $\quad |x_i|, |y_i| \le 2M + 4 + 2\sup\{|h|\colon h \in H(f)\}, \quad |x_i - y_i| \le 4\delta, \quad i = 1, 2,$

the following relations hold:

(4.50)
$$|U^f(0, 1, x_1, x_2) - U^f(0, 1, y_1, y_2)| \le 2^{-8}\delta_0,$$
$$|\pi^f(x_i) - \pi^f(y_i)| \le 2^{-8}\delta_0, \quad i = 1, 2.$$

By Proposition 2.5 there exists an integer $L \ge 1$ such that for each (f)-good function $\nu\colon [0, \infty) \to R^n$

(4.51) $$\operatorname{dist}(H(f), \{\nu(t)\colon t \in [T, T + L]\}) \le 8^{-1}\delta$$

for all large T.

By Proposition 2.8 there exists an integer $N \ge 10$ and a neighborhood $\mathcal{U}_2$ of f in $\mathfrak{A}$ such that for each $g \in \mathcal{U}_2$, each $S \in [0, \infty)$ and each a.c. function $x\colon [S, S+NL] \to R^n$ which satisfies

(4.52)
$$|x(S)|, |x(S + NL)| \le 2M + 2,$$
$$I^g(S, S + NL, x) \le U^g(S, S + NL, x(S), x(S + NL)) + 4$$

there exists an integer $i_0 \in [0, N - 8]$ such that for all $T \in [S + i_0 L, S + (i_0 + 7)L]$

(4.53) $$\operatorname{dist}(H(f), \{x(t)\colon t \in [T, T + L]\}) \le \delta.$$

By Lemma 4.4 there exists a number $q \geq 8$ such that for each $h_1, h_2 \in H(f)$ there exists an a.c. function $\nu \colon [0, q] \to R^n$ which satisfies

(4.54)
$$\nu(0) = h_1, \quad v(q) = h_2, \quad I^f(0, q, \nu) - q\mu(f) - \pi^f(\nu(0)) + \pi^f(\nu(q)) \leq 8^{-1}\delta.$$

By Proposition 2.3 there exists a neighborhood $\mathcal{U}_3$ of f in $\mathfrak{A}$ such that for each $g \in \mathcal{U}_3$, each $T_1 \geq 0$, $T_2 \in [T_1 + 8^{-1}, T_1 + 6N(q + l + L)]$ and each a.c. function $x \colon [T_1, T_2] \to R^n$ which satisfies

$$\inf\{I^f(T_1, T_2, x), I^g(T_1, T_2, x)\}$$

(4.55)
$$\leq 4 + 2\sup\{|\pi^f(h)| \colon h \in R^n, |h| \leq \sup\{|z| \colon z \in H(f)\} + 4\}$$
$$+ 6|\mu(f)|N(q + l + L)$$

the following relation holds

(4.56)
$$|I^f(T_1, T_2, x) - I^g(T_1, T_2, x)| \leq 4^{-1}\delta.$$

By Proposition 2.4 there exists a neighborhood $\mathcal{U}_4$ of f in $\mathfrak{A}$ such that for each $g \in \mathcal{U}_4$, each $x_1, x_2 \in R^n$ which satisfy

$$|x_1|, |x_2| \leq 2M + 4 + 2\sup\{|z| \colon z \in H(f)\}$$

the following relation holds:

(4.57)
$$|U^f(0, 1, x_1, x_2) - U^g(0, 1, x_1, x_2)| \leq 2^{-8}\delta.$$

Set

(4.58)
$$l_0 = 2l + 2q + 2NL + 6,$$

(4.59)
$$\mathcal{U} = \bigcap_{i=1}^{4} \mathcal{U}_i.$$

Assume that $g \in \mathcal{U}$, $T \geq 2l_0$ and an a.c. function $\nu \colon [0, T] \to R^n$ satisfies

(4.60)
$$|\nu(0)|, |\nu(T)| \leq K, \quad I^g(0, T, \nu) \leq U^g(0, T, \nu(0), \nu(T)) + \delta.$$

It follows from the definition of $\mathcal{U}_1$ (see (4.38), (4.39)) and (4.60) that

(4.61)
$$|\nu(t)| \leq M, \quad t \in [0, T].$$

Assume that there exist numbers $S_1, S_2 \in [0, T]$ such that

(4.62)
$$d(\nu(S_i), H(f)) \leq \delta, \quad i = 1, 2, \quad S_2 - S_1 \in [1 + l + q, 5N(L + l + q)].$$

We will show that for each $\tau \in [S_1, S_2 - l]$

(4.63)
$$\mathrm{dist}(H(f), \{\nu(t) \colon t \in [\tau, \tau + l]\}) \leq \varepsilon.$$

Let us assume the converse. Then there exists a number τ such that

(4.64)
$$\tau \in [S_1, S_2 - l], \quad \mathrm{dist}(H(f), \{\nu(t) \colon t \in [\tau, \tau + l]\}) > \varepsilon.$$

It follows from (4.62), (4.64), (4.48) and the definition of δ_0 (see (4.44)–(4.47)) that

(4.65)
$$I^f(S_1, S_2, \nu) - (S_2 - S_1)\mu(f) - \pi^f(\nu(S_1)) + \pi^f(\nu(S_2)) > \delta_0.$$

We will show that

(4.66)
$$I^g(S_1, S_2, \nu) - (S_2 - S_1)\mu(f) - \pi^f(\nu(S_1)) + \pi^f(\nu(S_2)) \geq 2^{-1}\delta_0.$$

Assume the contrary. Then by (4.62)

$$I^g(S_1, S_2, \nu) \leq 2\sup\{|\pi^f(z)| \colon z \in R^n, d(z, H(f)) \leq 1\} + |\mu(f)|(S_2 - S_1) + 1.$$

It follows from this relation, (4.62) and the definition of $\mathcal{U}_3$ (see (4.55), (4.56)) that

$$|I^f(S_1, S_2, \nu) - I^g(S_1, S_2, \nu)| \leq 4^{-1}\delta.$$

Together with (4.65) this implies (4.66). The obtained contradiction proves that (4.66) holds.

By (4.62) there exist $h_1, h_2 \in H(f)$ such that

$$(4.67) \qquad\qquad |\nu(S_i) - h_i| \leq \delta, \quad i = 1, 2.$$

By Lemma 4.1 there exists an (f)-good function

$$(4.68) \qquad\qquad w_0 \colon [0, \infty) \to H(f) \quad \text{such that} \quad w_0(0) = h_1,$$

$$(4.69) \qquad I^f(t_1, t_2, w_0) - (t_2 - t_1)\mu(f) - \pi^f(w_0(t_1)) + \pi^f(w_0(t_2)) = 0$$

for each $t_1 \geq 0$, $t_2 > t_1$.

It follows from the definition of q (see (4.54), (4.62), (4.68)) that there exists an a.c. function $w_1 \colon [0, q] \to R^n$ such that

$$(4.70) \qquad \begin{aligned} w_1(0) &= w_0(S_2 - S_1 - q), \quad w_1(q) = h_2, \\ I^f(0, q, w_1) &- q\mu(f) - \pi^f(w_1(0)) + \pi^f(w_1(q)) \leq 8^{-1}\delta. \end{aligned}$$

By (4.62), Proposition 2.13, (4.68), (4.70) there exists an a.c. function $u \colon [0, T] \to R^n$ such that

$$(4.71) \qquad \begin{aligned} u(t) &= \nu(t), \quad t \in [0, S_1] \cup [S_2, T], \\ u(t) &= w_0(t - S_1), \quad t \in [S_1 + 1, S_2 - q], \\ u(t) &= w_1(t - (S_2 - q)), \quad t \in [S_2 - q, S_2 - 1], \\ I^g(r, r + 1, u) &= U^g(0, 1, u(r), u(r + 1)), \quad r = S_1, S_2 - 1. \end{aligned}$$

For each a.c. function $y \colon [a, b] \to R^n$ where $a \geq 0$, $b > a$ and each $r_1, r_2 \in [a, b]$ satisfying $r_1 \leq r_2$ we set

$$(4.72) \qquad \sigma(r_1, r_2, y) = I^g(r_1, r_2, y) - \pi^f(y(r_1)) + \pi^f(y(r_2)) - (r_2 - r_1)\mu(f).$$

It follows from (4.60), (4.71), (4.72) that

$$(4.73) \qquad \begin{aligned} \delta &\geq I^g(0, T, \nu) - I^g(0, T, u) \\ &= \sigma(0, T, \nu) - \sigma(0, T, u) = \sigma(S_1, S_2, \nu) - \sigma(S_1, S_2, u). \end{aligned}$$

By (4.70), (4.68) and the definition of M (see (4.38), (4.39))

$$(4.74) \qquad\qquad |w_1(t)| \leq M, \quad t \in [0, q].$$

By (4.70) there exists an a.c. function $\tilde{w} \colon [S_1, S_2] \to R^n$ such that

$$(4.75) \qquad \begin{aligned} \tilde{w}(t) &= w_0(t - S_1), \quad t \in [S_1, S_2 - q], \\ \tilde{w}(t) &= w_1(t - (S_2 - q)), \quad t \in [S_2 - q, S_2]. \end{aligned}$$

(4.73), (4.72), (4.66), (4.71), (4.75) imply that

$$(4.76) \qquad \begin{aligned} \delta &\geq 2^{-1}\delta_0 - \sigma(S_1, S_2, u) = 2^{-1}\delta_0 - \sigma(S_1, S_2, \tilde{w}) \\ &+ [\sigma(S_1, S_1 + 1, \tilde{w}) - \sigma(S_1, S_1 + 1, u)] \\ &+ [\sigma(S_2 - 1, S_2, \tilde{w}) - \sigma(S_2 - 1, S_2, u)]. \end{aligned}$$

We will estimate $\sigma(S_1, S_2, \tilde{w})$ and $\sigma(h, h + 1, \tilde{w}) - \sigma(h, h + 1, u)$, $h = S_1, S_2 - 1$.

Let $h \in \{S_1, S_2 - 1\}$. (4.70), (4.75), (4.71), (4.74), (4.68), (4.61), (4.62), (4.67) imply that

$$|\tilde{w}(h)|, |\tilde{w}(h+1)|, |u(h)|, |u(h+1)| \leq M + \sup\{|z| \colon z \in H(f)\},$$
$$|\tilde{w}(h) - u(h)|, |\tilde{w}(h+1) - u(h+1)| \leq \delta.$$

It follows from these relations, (4.71), (4.72), the definition of $\mathcal{U}_4$ (see (4.57)) and δ (see (4.49), (4.50), (4.48)) that

$$
\begin{aligned}
(4.77) \quad \sigma(h, h+1, \tilde{w}) &- \sigma(h, h+1, u) \\
&\geq U^g(0, 1, \tilde{w}(h), \tilde{w}(h+1)) - \pi^f(\tilde{w}(h)) + \pi^f(\tilde{w}(h+1)) \\
&\quad - [U^g(0, 1, u(h), u(h+1)) - \pi^f(u(h)) + \pi^f(u(h+1))] \\
&\geq U^f(0, 1, \tilde{w}(h), \tilde{w}(h+1)) - \pi^f(\tilde{w}(h)) + \pi^f(\tilde{w}(h+1)) \\
&\quad - [U^f(0, 1, u(h), u(h+1)) - \pi^f(u(h)) + \pi^f(u(h+1))] - 2^{-7}\delta \\
&\geq -2^{-6}\delta_0, \quad h \in \{S_1, S_2 - 1\}.
\end{aligned}
$$

We will estimate $\sigma(S_1, S_2, \tilde{w})$. It follows from (4.69), (4.70), (4.75) that

$$(4.78) \qquad I^f(S_1, S_2, \tilde{w}) - \pi^f(\tilde{w}(S_1)) + \pi^f(\tilde{w}(S_2)) - (S_2 - S_1)\mu(f) \leq 8^{-1}\delta.$$

By this relation, (4.62), (4.75), (4.68), (4.70) and the definition of $\mathcal{U}_3$ (see (4.55), (4.56))

$$|I^f(S_1, S_2, \tilde{w}) - I^g(S_1, S_2, \tilde{w})| \leq 4^{-1}\delta.$$

Together with (4.78), (4.72) this implies that

$$\sigma(S_1, S_2, \tilde{w}) \leq 3 \cdot 8^{-1}\delta.$$

It follows from this relation, (4.76), (4.77) that

$$\delta \geq 2^{-1}\delta_0 - 3 \cdot 8^{-1}\delta - 2^{-5}\delta_0.$$

This is contradictory to (4.48). The obtained contradiction proves that (4.63) holds for each $\tau \in [S_1, S_2 - l]$. Therefore we have shown that the following property holds:

Property D. For each $S_1, S_2 \in [0, T]$ which satisfy (4.62) relation (4.63) holds for each $\tau \in [S_1, S_2 - l]$.

It follows from (4.60), (4.61) and the definition of $\mathcal{U}_2, N$ (see (4.52), (4.53)) that for each $r_0 \in [0, T - (1 + l + q + L(N + 2))]$ there exists a number r_1 such that

$$r_1 - r_0 \in [1 + l + q + 2L, 1 + l + q + L(N + 2)], \quad d(\nu(r_1), H(f)) \leq \delta.$$

This implies that there exists a finite sequence of numbers $\{S_i\}_{i=0}^{Q} \subset [0, T]$ such that

$$
\begin{aligned}
S_0 = 0, \, S_{i+1} - S_i &\in [1 + l + q + 2L, 1 + l + q + L(N + 2)], \quad i = 0, \ldots, Q - 1, \\
T - S_Q &\leq 1 + l + q + L(N + 2), \quad d(\nu(S_i), H(f)) \leq \delta, \quad i = 1, \ldots, Q.
\end{aligned}
$$

The assertion of the theorem follows from these relations and Property D.

5. Proof of Theorem 1.3

Lemma 5.1. *Assume that an integrand $f \in \mathfrak{M}$ has Property B and $\varepsilon > 0$. Then there exists a neighborhood $\mathcal{U}$ of f in $\mathfrak{A}$ such that for each $g \in \mathcal{U}$ and each (g)-good function $\nu \colon [0, \infty) \to R^n$*

$$\mathrm{dist}(\Omega(\nu), H(f)) \leq \varepsilon.$$

Proof. By Proposition 2.1 there exist a neighborhood $\mathcal{U}_1$ of f in $\mathfrak{A}$ and a number $K > 0$ such that for each $g \in \mathcal{U}_1$ and each (g)-good function $\nu \colon [0, \infty) \to R^n$

$$\limsup_{t \to \infty} |\nu(t)| < K.$$

By Theorem 1.2 there exist a neighborhood $\mathcal{U}$ of f in $\mathfrak{A}$ which satisfies $\mathcal{U} \subset \mathcal{U}_1$ and numbers $l_0 > l > 0$, $\delta > 0$ such that for each $g \in \mathcal{U}$, each $T \geq 2l_0$ and each a.c. function $\nu \colon [0, T] \to R^n$ which satisfies

$$|\nu(0)|, |\nu(T)| \leq K, \quad I^g(0, T, \nu) \leq U^g(0, T, \nu(0), \nu(T)) + \delta$$

the relation $\mathrm{dist}(H(f), \{\nu(t) \colon t \in [\tau, \tau + l]\}) \leq \varepsilon$ holds for each $\tau \in [l_0, T - l_0]$.

Assume that $g \in \mathcal{U}$ and $v \colon [0, \infty) \to R^n$ is a (g)-good function. It follows from the definition of $\mathcal{U}$, $\mathcal{U}_1$, K and Proposition 1.1 that there exists a number $T_0 \geq 0$ such that

$$|\nu(t)| \leq K, \quad t \in [T_0, \infty),$$
$$I^g(t_1, t_2, \nu) \leq U^g(t_1, t_2, \nu(t_1), \nu(t_2)) + \delta \quad \text{for each } t_1 \geq T_0, \ t_2 > t_1.$$

It follows from these relations and the definition of $\mathcal{U}$, l_0, l, δ that

$$\mathrm{dist}(H(f), \Omega(\nu)) \leq \varepsilon.$$

The lemma is proved.

Construction of the set $\mathcal{F}_q$. Suppose that q is a nonnegative integer. By Propositions 2.11, 2.12, 2.15 there exists a set $E_q \subset \mathfrak{M}_q$ which is an everywhere dense subset of $\overline{\mathfrak{M}}_q$ and such that each integrand $f \in E_q$ has Property B. Therefore for each $f \in E_q$ there exists a compact set $H(f) \subset R^n$ such that $\Omega(\nu) = H(f)$ for each (f)-good function $\nu \colon [0, \infty) \to R^n$.

By Theorem 1.2 and Lemma 5.1 for each $f \in E_q$ and each integer $p \geq 1$ there exist an open neighborhood $\mathcal{U}(f, p)$ of f in $\mathfrak{A}$ and numbers $M(f, p) > p$, $l_0(f, p) > l(f, p) > 0$, $\delta(f, p) \in (0, p^{-1})$ such that for each $g \in \mathcal{U}(f, p)$ and each (g)-good function $\nu \colon [0, \infty) \to R^n$

$$\mathrm{dist}(H(f), \Omega(\nu)) \leq 4^{-1}\delta(f, p);$$

for each $g \in \mathcal{U}(f, p)$, each $T \geq 2l_0(f, p)$ and each a.c. function $\nu \colon [0, T] \to R^n$ which satisfies

$$(5.1) \qquad |\nu(0)|, |\nu(T)| \leq p, \quad I^g(0, T, \nu) \leq U^g(0, T, \nu(0), \nu(T)) + \delta(f, p)$$

the relation $|\nu(t)| \leq M(f, p)$ holds for all $t \in [0, T]$ and the following properties hold:

(i) for each $\tau \in [l_0(f, p), T - l_0(f, p)]$

$$(5.2) \qquad \mathrm{dist}(H(f), \{\nu(t) \colon t \in [\tau, \tau + l(f, p)]\}) \leq p^{-1};$$

(ii) if $d(\nu(0), H(f)) \leq \delta(f, p)$, then (5.2) holds for each $\tau \in [0, T - l_0(f, p)]$;
(iii) if $d(\nu(T), H(f)) \leq \delta(f, p)$, then (5.2) holds for each $\tau \in [l_0(f, p), T - l(f, p)]$.

We define

$$(5.3) \qquad \mathcal{F}_q = \left[\bigcap_{p=1}^{\infty} \cup \{ \mathcal{U}(f,p) \colon f \in E_q \} \right] \cap \overline{\mathfrak{M}}_q.$$

Clearly $\mathcal{F}_q$ is a countable intersection of open everywhere dense subsets of $\overline{\mathfrak{M}}_q$.

Assume that $f \in \mathcal{F}_q$, $\varepsilon, K > 0$. Fix a natural number p such that

$$(5.4) \qquad p > 2K + 4 + 8\varepsilon^{-1}.$$

There exists $G \in E_q$ such that

$$(5.5) \qquad f \in \mathcal{U}(G,p).$$

It follows from (5.4), (5.5) and the definition of $\mathcal{U}(G,p)$, $\delta(G,p)$ that for each (f)-good function $\nu \colon [0,\infty) \to R^n$

$$(5.6) \qquad \mathrm{dist}(H(G), \Omega(\nu)) \leq 4^{-1}\delta(G,p) < (4p)^{-1} < 8^{-1}\varepsilon.$$

This implies that for each (f)-good function $\nu_i \colon [0,\infty) \to R^n$, $i = 1, 2$,

$$\mathrm{dist}(\Omega(\nu_1), \Omega(\nu_2)) \leq \varepsilon.$$

Since ε is any positive number we conclude that f has Property B and there exists a compact set $H(f) \subset R^n$ such that $\Omega(w) = H(f)$ for each (f)-good function $w \colon [0,\infty) \to R^n$. It follows from (5.6) that

$$(5.7) \qquad \mathrm{dist}(H(G), H(f)) \leq 4^{-1}\delta(G,p).$$

Set

$$(5.8) \qquad \begin{aligned} \mathcal{U} &= \mathcal{U}(G,p), \quad M = M(G,p), \\ l_0 &= l_0(G,p), \quad l = l(G,p), \quad \delta = 8^{-1}\delta(G,p). \end{aligned}$$

Assume that $g \in \mathcal{U}$, $T \geq 2l_0$ and an a.c. function $\nu \colon [0,T] \to R^n$ satisfies

$$(5.9) \qquad |\nu(0)|, |\nu(T)| \leq K, \quad I^g(0,T,\nu) \leq U^g(0,T,\nu(0),\nu(T)) + \delta.$$

It follows from (5.9), (5.8), (5.5), (5.4) and the definition of $\mathcal{U}(G,p)$, $M(G,p)$, $l_0(G,p)$, $l(G,p)$, $\delta(G,p)$ that

$$(5.10) \qquad |\nu(t)| \leq M, \quad t \in [0,T],$$

and properties (i)–(iii) hold with $f = G$. Together with (5.7), (5.8), (5.4) this implies that

$$(5.11) \qquad \mathrm{dist}(H(f), \{\nu(t) \colon t \in [\tau, \tau + l]\}) \leq \varepsilon$$

for each $\tau \in [l_0, T - l_0]$; if $d(\nu(0), H(f)) \leq \delta$, then (5.11) holds for each $\tau \in [0, T - l_0]$. If $d(\nu(T), H(f)) \leq \delta$, then (5.11) holds for each $\tau \in [l_0, T - l]$. This completes the proof of the theorem.

6. EXAMPLES

Fix a constant $a > 0$ and set $\psi(t) = t$, $t \in [0,\infty)$. Consider the complete metric space $\mathfrak{A}$ of integrands $f \colon R^n \times R^n \to R^1$ defined in Section 1.

Example 1. Consider an integrand $f(x,u) = |x|^2 + |u|^2$, $x, u \in R^n$. It is easy to see that $f \in \mathfrak{M}_q$ for each integer $q \geq 0$ if the constant a is large enough. We can show (see [20, Section 14]) that $\Omega(\nu) = \{0\}$ for every (f)-good function $\nu \colon [0,\infty) \to R^n$. Therefore Theorem 1.2 holds with the integrand f.

Example 2. Fix a number $q > 0$ and consider an integrand

$$g(x, u) = q|x|^2|x - e|^2 + |u|^2, \quad x, u \in R^n,$$

where $e = (1, 1, \ldots, 1)$. It is easy to see that $g \in \mathfrak{M}$ if the constant a is large enough. Clearly f does not have the turnpike property.

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